

A SURVEY ON RANKIN-COHEN DEFORMATIONS

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ABSTRACT. This is a survey about recent progress in Rankin-Cohen deformations. We explain a connection between Rankin-Cohen brackets and higher order Hankel forms.

1. INTRODUCTION

The famous *Erlanger Programm* of Klein says that geometry is about to study the transformation groups of various spaces, or more precisely the properties invariant under the actions of such groups, *i.e.*, the symmetries.

Noncommutative geometry (NCG), which originated from Connes' study in operator algebras in 1970's, brought the landscape of geometry many new objects and some astonishing phenomena.

Back to the early 1990s, Connes and Moscovici pointed out that in noncommutative geometry (NCG) while noncommutative spaces are represented by the algebras (usually noncommutative C^* -algebras) of “continuous functions” over noncommutative spaces, the *local* symmetries are reflected in some Hopf algebras. One of the first noncommutative spaces studied in NCG is the C^* -algebra of a foliated space. In the case of codimension n foliations, Connes and Moscovici discovered a Hopf algebra \mathcal{H}_n , which governs the local symmetry of leaf spaces of foliations of codimension n . The Hopf algebra \mathcal{H}_n is universal in the sense that it depends only on the codimension of a foliation. This family of Hopf algebras $\{\mathcal{H}_n\}$ is very useful in the study of transverse index theory, and later was found to have connections with various different areas of mathematics, c.f. [10], [13]. In this paper, we review the application of the Hopf algebra \mathcal{H}_1 in Rankin-Cohen deformations, which was initiated by Connes and Moscovici [14].

We start by recalling the general setting of transverse geometry. Let M be a smooth manifold and \mathcal{F} be a foliation on M of codimension n . Let X be a complete flat transversal of \mathcal{F} , and F^+X be the oriented frame bundle of X . The holonomy pseudogroup Γ acts on X and therefore F^+X by transforming X parallelly along paths in leaves of \mathcal{F} . The “transverse geometry” is to study the transversal X along with the action by the holonomy pseudogroup Γ .

In what follows we focus on the case when $n = 1$, and define Connes-Moscovici's Hopf algebra \mathcal{H}_1 . Now the complete transversal X is a flat 1-dim manifold; and the oriented frame bundle F^+X is diffeomorphic to $X \times \mathbb{R}^+$, and Γ is a discrete holonomy pseudogroup acting on X as local diffeomorphisms. We introduce coordinates x on X and y on \mathbb{R}^+ . The lifted action of Γ on F^+X is

$$(x, y) \mapsto (\phi(x), \phi'(x)y), \quad \phi \in \Gamma. \quad (1.1)$$

On F^+X , there is a Γ -invariant volume form $\omega = \frac{dx \wedge dy}{y^2}$, which is also a symplectic form. This allows to consider the Hilbert space $L^2\left(F^+X, \frac{dx \wedge dy}{y^2}\right)$ of square-integrable functions on F^+X . We are interested in two types of linear operators acting on this Hilbert space $L^2\left(F^+X, \frac{dx \wedge dy}{y^2}\right)$:

- (1) for $f \in C_c^\infty(F^+X)$, we define $M_f : \xi \mapsto f\xi$ for all $\xi \in L^2\left(F^+X, \frac{dx \wedge dy}{y^2}\right)$;
- (2) for $\phi \in \Gamma$, we define $U_\phi : \xi \mapsto \phi^*(\xi) = \xi \circ \phi$ for all $\xi \in L^2\left(F^+X, \frac{dx \wedge dy}{y^2}\right)$.

The smooth foliation algebra \mathcal{A}_Γ is the algebra generated by M_f 's and U_ϕ 's with the relation

$$U_\phi M_f = M_{\phi^*(f)} U_\phi.$$

From this relation, we can say that \mathcal{A}_Γ is the cross product algebra $C_c^\infty(F^+X) \rtimes \Gamma$. In noncommutative geometry, this algebra \mathcal{A}_Γ is viewed as the algebra of smooth functions on the space of leaves associated to the foliation \mathcal{F} on M . The Hopf algebra \mathcal{H}_1 acts on the smooth foliation algebra by linear operators.

By choosing a flat connection on F^+X , we consider two vector fields, i) $X = y\partial_x$ as a lifting of the vector field ∂_x on X , and ii) $Y = y\partial_y$ the Euler vector field along the fiber direction. X and Y act on the smooth foliation algebra \mathcal{A}_Γ by

$$X(fU_\phi) = X(f)U_\phi, \quad Y(fU_\phi) = Y(f)U_\phi.$$

It is easy to check that Y is invariant under the action of Γ , but X is not:

$$U_\phi X U_\phi^{-1} = X - y \frac{\phi^{-1''}(x)}{\phi^{-1'}(x)} Y.$$

The failure of X being Γ invariant inspires higher operations. Define a linear operator δ_1 on \mathcal{A}_Γ in the following way:

$$\delta_1(fU_\phi) = \mu_{\phi^{-1}} fU_\phi,$$

where $\mu_{\phi^{-1}}(x, y) = y \frac{\phi^{-1''}(x)}{\phi^{-1'}(x)}$.

We compute $[Y, \delta_1]$, which turns out to be δ_1 itself. But $[X, \delta_1]$ leads to a new operator, which we name δ_2 :

$$\delta_2(fU_\phi) = X(\mu_{\phi^{-1}}) fU_\phi.$$

Iterating the procedure of computing the commutator with X , we are led to a sequence of operators δ_n acting on \mathcal{A}_Γ by

$$\delta_n(fU_\phi) = X^{n-1}(\mu_{\phi^{-1}}) fU_\phi.$$

With the above preparation, we are ready to present the Hopf algebra \mathcal{H}_1 . As an algebra, it is the universal enveloping algebra of an infinite dimensional Lie algebra H_1 whose generators are labeled as $\{X, Y, \delta_n, n \in \mathbb{N}\}$ with the following commutation relations:

$$\begin{aligned} [Y, X] &= X, & [X, \delta_n] &= \delta_{n+1}, \\ [Y, \delta_n] &= n\delta_n, & [\delta_n, \delta_m] &= 0. \end{aligned}$$

We define the following structures on \mathcal{H}_1 :

- (1) product $\cdot : \mathcal{H}_1 \otimes \mathcal{H}_1 \rightarrow \mathcal{H}_1$ is defined as the product on the universal enveloping algebra of H_1 .

(2) coproduct $\Delta : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_1$ is an algebra homomorphism generated by

$$\begin{aligned}\Delta Y &= Y \otimes 1 + 1 \otimes Y, \\ \Delta \delta_1 &= \delta_1 \otimes 1 + 1 \otimes \delta_1, \\ \Delta X &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \\ \Delta \delta_n &= [\Delta X, \Delta \delta_{n-1}].\end{aligned}$$

(3) counit $\epsilon : \mathcal{H}_1 \rightarrow \mathbb{C}$ is defined by taking the constant component in \mathcal{H}_1 .

(4) antipode $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ is an algebra anti-homomorphism generated by

$$S(X) = -X + \delta_1 Y, \quad S(Y) = -Y, \quad S(\delta_1) = -\delta_1.$$

It is straightforward to check that $(\mathcal{H}_1, \cdot, \Delta, S, \epsilon, id)$ defines a Hopf algebra, which acts naturally on the smooth foliation algebra \mathcal{A}_Γ .

The Hopf algebra \mathcal{H}_1 and its higher dimensional generalizations were discovered by Connes and Moscovici [11] as local symmetries on \mathcal{A}_Γ . In the following, we will survey some recent developments about the applications of \mathcal{H}_1 in the study of modular forms and in particular Rankin-Cohen deformations.

The results reviewed in this paper are interactions between transverse geometry and modular form theory. These two classical “distant” subjects are mysteriously connected due to the fact that the Hopf algebra \mathcal{H}_1 appears in both theories as the local symmetry of the corresponding systems. What we will develop in the last section of this paper is to add one more subject to this story, namely the Hankel forms and transvectant theory in harmonic analysis. There, we will introduce an algebra \mathcal{B}_Γ associated to a pseudogroup Γ acting on a 1-dim complex domain by holomorphic transformations. We will show that the Hopf algebra \mathcal{H}_1 acts on \mathcal{B}_Γ . And interestingly, through the Hopf algebra action the Rankin-Cohen brackets on modular form are translated to Hankel forms of higher weights [21].

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2. RANKIN-COHEN BRACKETS AND DEFORMATIONS

In number theory, modular forms are very important because the coefficients of their Fourier expansions encode a great amount of number theoretical information. We recall the definition of a modular form. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ (a subgroup of $SL_2(\mathbb{R})$ such that the entries of a matrix are all integers), a modular form of weight $2k$ is a function f which satisfies:

- (holomorphy) f is holomorphic on the upper-half plane \mathbb{H} ,
- (modularity) for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathbb{H}$, $f|_{2k} \gamma = f$, where

$$\left(f|_{2k} \gamma\right)(z) = (cz + d)^{-2k} f\left(\frac{az + b}{cz + d}\right),$$

(this means the invariance of the form $f dz^k$);

- (growth condition at the boundary) $|f(z)|$ is assumed to be controlled by a polynomial in $\max\{1, \text{Im}(z)^{-1}\}$.

We denote by $\mathcal{M}(\Gamma)$ the (weight) graded algebra of modular forms with respect to the group Γ .

The Rankin-Cohen brackets are a family of universal formulas describing all bilinear operations (up to a scalar) between spaces of modular forms (of even weight) that can be defined in terms of derivatives. In the 50's Rankin started the research on bi-differential operators on $\mathcal{M}(\Gamma)$ which produce new modular forms, and twenty years later Heri Cohen¹ gave a complete answer (cf. [5]) to this question by showing that all these operators are linear combinations of the bracket

$$[f, g]_n = \sum_{r=0}^n (-1)^r \binom{n+2k-1}{n-r} \binom{n+2l-1}{r} f^{(r)} g^{(n-r)} \in \mathcal{M}_{2k+2l+2n}(\Gamma), \quad (2.1)$$

where $f \in \mathcal{M}_{2k}$ and $g \in \mathcal{M}_{2l}$ are two modular forms, and $f^{(r)} = \left(\frac{1}{2\pi i} \frac{\partial}{\partial z} \right)^r f$. The above bilinear operation $[,]_n$ is called n -th Rankin-Cohen bracket.

The original and obvious importance of the Rankin-Cohen brackets in number theory is that these brackets often give rise to non-trivial identities between the Fourier coefficients of modular forms, e.g. H. Cohen's foundational paper [5]. Zagier [32] has found several other "reasons d'être" for the Rankin-Cohen brackets. He discovered the following procedure to obtain the Rankin-Cohen brackets: we denote Ramanujan's derivation on modular forms by $X : \mathcal{M}_{2k}(\Gamma) \rightarrow \mathcal{M}_{2k+2}(\Gamma)$:

$$\begin{aligned} Xf &= \frac{1}{2\pi i} \frac{df}{dz} - \frac{1}{2\pi i} \frac{\partial}{\partial z} (\log \eta^4) \cdot kf, \\ \eta(z) &= q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi iz}, \end{aligned}$$

and define two sequences of modular forms by induction: for $\Phi \in \mathcal{M}_4(\Gamma)$, $f_0 = f, g_0 = g$,

$$\begin{aligned} f_{r+1} &= Xf_r + r(r+2k-1)\Phi f_{r-1}, \\ g_{s+1} &= Xg_s + s(s+2l-1)\Phi g_{s-1}. \end{aligned} \quad (2.2)$$

Zagier showed [32] that the Rankin-Cohen brackets can be written as

$$\sum_{r=0}^n (-1)^r \binom{n+2k-1}{n-r} \binom{n+2l-1}{r} f_r g_{n-r} = [f, g]_n.$$

These forms of the brackets have the advantage that one can easily see the modularity of $[f, g]_n$ without any extra effort from the definition of f_r and g_s , while in the original presentation, the modularity of the brackets is far from being obvious as the derivative of a modular form is in general not a modular form any more.

Zagier and his collaborators [6] showed that the collection of all Rankin-Cohen brackets together gives rise to (non-commutative) associative deformations of the algebra $\mathcal{M}(\Gamma)$ of modular forms. We review their constructions in more detail.

Paula Cohen², Manin, and Zagier [6] established a bijection between formal series with modular form coefficients and formal invariant pseudodifferential operators. For a modular

¹We have given the first name of the author because in the literature of Rankin-Cohen deformations, there are two different and important authors with the same last name, "Cohen".

²Here is the other author with the same last name "Cohen".

form f of weight $2k$ we define

$$\mathcal{D}_{-k}(f, \partial) = \sum_{n=0}^{\infty} \frac{\binom{-k}{n} \binom{-k-1}{n}}{\binom{-2k}{n}} f^{(n)} \partial^{-k-n}, \quad (2.3)$$

and the composition of two such formal pseudodifferential operators (plus the bilinearity) give us the following product:

$$f \star g = \sum_{n=0}^{\infty} c_n(k, l) [f, g]_n, \quad (2.4)$$

$$c_n(k, l) = t_n(k, l) := \frac{1}{\binom{-2l}{n}} \sum_{r+s=n} \frac{\binom{-k}{r} \binom{-k-1}{r}}{\binom{-2k}{r}} \frac{\binom{n+k+l}{s} \binom{n+k+l-1}{s}}{\binom{2n+2k+2l-2}{s}}. \quad (2.5)$$

Meanwhile, by checking the first several terms, Eholzer [32] conjectured that

$$f \star g := \sum_{n=0}^{\infty} [f, g]_n \quad (2.6)$$

is also an associative product.

In order to include this phenomenon in their framework, P. Cohen, Manin, and Zagier [6] modified their definition of invariant formal pseudodifferential operators, and they finally obtained a whole family of such deformations, parametrized by the Riemann sphere([6],[32]). In this most general case, we consider $\gamma \in SL_2(\mathbb{C})$ action on functions $\xi \in C^\infty(\mathbb{H})$:

$$(W_\gamma^\kappa \xi)(z) = \xi\left(\frac{az+b}{cz+d}\right) (cz+d)^\kappa.$$

Then we can define

$$\mathcal{D}_{-k}^\kappa(f, \partial) = \sum_{n=0}^{\infty} \frac{\binom{-k}{n} \binom{-k+\kappa-1}{n}}{\binom{-2k}{n}} f^{(n)} \partial^{-k-n}, \quad (2.7)$$

which has the following invariant property:

$$\mathcal{D}_{-k}^\kappa(f|_{2k\gamma}, \partial) = W_\gamma^{\kappa*} \mathcal{D}_{-k}^\kappa(f, \partial) W_\gamma^\kappa.$$

By composing two such formal invariant pseudodifferential operators, we obtain, over the algebra of formal series with coefficients in modular forms $\mathcal{M}[[\hbar]]$, that the natural linear extension plus the following formula gives an associative product: for $f \in \mathcal{M}_{2k}$, $g \in \mathcal{M}_{2l}$,

$$\mu^\kappa(f, g) = \sum_{n=0}^{\infty} t_n^\kappa(k, l) [f, g]_n \hbar^n, \quad (2.8)$$

where the coefficients are

$$t_n^\kappa(k, l) = \frac{1}{\binom{-2l}{n}} \sum_{r+s=n} \frac{\binom{-k}{r} \binom{-k+\kappa-1}{r}}{\binom{-2k}{r}} \frac{\binom{n+k+l-\kappa}{s} \binom{n+k+l-1}{s}}{\binom{2n+2k+2l-2}{s}}. \quad (2.9)$$

Moreover, the coefficients $t_n^\kappa(k, l)$ are conjectured to be equal to

$$t_n^\kappa(k, l) = \left(-\frac{1}{4}\right)^n \sum_{j \geq 0} \binom{n}{2j} \frac{\binom{-\frac{1}{2}}{j} \binom{\kappa - \frac{3}{2}}{j} \binom{\frac{1}{2} - \kappa}{j}}{\binom{-k - \frac{1}{2}}{j} \binom{-l - \frac{1}{2}}{j} \binom{n + k + l - \frac{3}{2}}{j}}. \quad (2.10)$$

By taking $\kappa = \frac{1}{2}$ or $\frac{3}{2}$ under this form, (1.3) turns to be Eholzer's product Eq. (2.4), and until 2004 this is the only possible way to prove the associativity of Eholzer's product. (Unfortunately, Zagier's original proof [34] of the identity Eq. (2.10) is not published, and one can find an elementary but rather long proof in [30]).

3. MODULAR HECKE ALGEBRAS AND CONNES-MOSCOVICI'S DEFORMATION

The interaction between the theory of modular forms and noncommutative geometry goes back to December 2001, when Zagier gave a course at Collège de France ([33]) and Connes was in the audience. One year later, Connes and Moscovici discovered that the Hopf algebra \mathcal{H}_1 that controls the local symmetry of the transverse geometry of codimension one foliations does act on some big algebra constructed from the algebra of modular forms, and named it “modular Hecke algebra”, which we will briefly recall now.

We first define

$$\mathcal{M}(\Gamma(N)) := \Sigma^\oplus \mathcal{M}_{2k}(\Gamma(N)), \quad \mathcal{M}^0(\Gamma(N)) := \Sigma^\oplus \mathcal{M}_{2k}^0(\Gamma(N)),$$

These algebras form a projective system with respect to the divisibility of the integer N . Define

$$\mathcal{M} := \varprojlim_{N \rightarrow \infty} \mathcal{M}(\Gamma(N)), \quad \text{resp.} \quad \mathcal{M}^0 := \varprojlim_{N \rightarrow \infty} \mathcal{M}^0(\Gamma(N)).$$

An *operator Hecke form of level Γ* [13] is a map

$$\begin{aligned} F : \Gamma \backslash GL_2^+(\mathbb{Q}) &\rightarrow \mathcal{M}, \\ \Gamma_\alpha &\mapsto F_\alpha \in \mathcal{M}, \end{aligned}$$

which has a finite support, and satisfies the *covariance condition*:

$$F_{\alpha\gamma}(z) = F_\alpha|\gamma(z), \quad \forall \alpha \in GL_2^+(\mathbb{Q}), \gamma \in \Gamma, z \in \mathbb{H}.$$

The modular Hecke algebra $\mathcal{A}(\Gamma)$ is an associative algebra consisting of operator Hecke forms of level Γ with the product,

$$(F^1 * F^2)_\alpha := \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\beta^1 \cdot F_{\alpha\beta^{-1}}^2 \Big| \beta.$$

An important discovery of Connes and Moscovici in [13] is that the Hopf algebra \mathcal{H}_1 acts on $\mathcal{A}(\Gamma)$.

Before we give the detail of this action, we briefly recall the general definition of a Hopf algebra action on an algebra. Let H be a Hopf algebra and M be an algebra. We say that $\alpha : H \otimes M \rightarrow M$ defines an action of H on M if the following two conditions hold;

- (1) M is an H -module with respect to the algebra structure on H ;
- (2) the following property holds with respect to the coalgebra structure on H ,

$$\alpha(h, a_1 a_2) = m((\alpha \otimes \alpha)(\Delta(h), a_1 \otimes a_2)),$$

where a_1, a_2 are elements of M , Δ is the coproduct of H , and $m : M \otimes M \rightarrow M$ is the multiplication operator.

With this definition, one can easily check that Connes-Moscovici's Hopf algebra \mathcal{H}_1 acts on the smooth foliation algebra \mathcal{A}_Γ . In the following, we review the \mathcal{H}_1 's action on modular Hecke algebra $\mathcal{A}(\Gamma)$ as was introduced by Connes-Moscovici [13].

For an $f \in \mathcal{M}_{2k}$, we define

$$Xf = \frac{1}{2\pi i} \frac{\partial}{\partial z} - \frac{1}{2\pi i} \frac{\partial}{\partial z} (\log \eta^4) \cdot kf, \quad Y(f) = k \cdot f.$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+(2, \mathbb{Q})$, we define

$$\mu_\gamma(z) = \frac{1}{2\pi^2} \left(G_2^*|_\gamma(z) - G_2^*(z) + \frac{2\pi i c}{cz + d} \right),$$

$$G_2^*(z) = 2\zeta(z) + 2 \sum_{m \geq 1} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2} = \frac{\pi^2}{3} - 8\pi^2 \sum_{m, n \geq 1} m e^{2\pi i m n z}.$$

We notice that here $\mu_\gamma \equiv 0$ if $\gamma \in SL_2(\mathbb{Z})$.

We define the action of X, Y, δ_n on $\mathcal{A}(\Gamma)$ as follows, for $F \in \mathcal{A}(\Gamma), \alpha \in G^+(\mathbb{Q})$,

$$\begin{aligned} X(F)_\alpha &:= X(F_\alpha), \\ Y(F)_\alpha &:= Y(F_\alpha), \\ \delta_n(F)_\alpha &:= \mu_{n, \alpha} \cdot F_\alpha, \text{ where } \mu_{n, \alpha} := X^{n-1}(\mu_\alpha), \quad n \in \mathbb{N}. \end{aligned}$$

With the above preparation, it is not difficult to check the following theorem.

Theorem 3.1. ([13]) *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$.*

- (1) *The Hopf algebra \mathcal{H}_1 acts on the algebra $\mathcal{A}(\Gamma)$.*
- (2) *The Schwarzian derivative $\delta'_2 = \delta_2 - \frac{1}{2}\delta_1^2$ is inner and is implemented by $\omega_4 = -\frac{1}{72}E_4 \in \mathcal{A}(\Gamma)$, where*

$$E_4 = 1 + 240 \sum_{n=1}^{\infty} n^3 \frac{e^{2\pi i n z}}{1 - e^{2\pi i n z}}$$

is an Eisenstein series of degree 4.

In fact, Connes and Moscovici [13] pointed out that $\mathcal{A}(\Gamma)$ can actually be obtained as the crossed product of the algebra of modular forms by the Hecke ring, and then reduced by a projection which is determined by the congruence subgroup Γ . Based on this fact, and assuming the associativity of the Ehlerz product, Connes and Moscovici [14] subsequently proved that this associative formal deformations can be canonically extended from the algebra of modular forms to the modular Hecke algebra associated to a congruence subgroup.

Remark 3.2. *Inspired by Theorem 3.1 (2), Connes and Moscovici introduced a concept of projectivity of an \mathcal{H}_1 action on an algebra \mathcal{A} as follows: there is an element $\Omega \in \mathcal{A}$ such that,*

$$(1) \quad \delta'_2(a) := \left(\delta_2 - \frac{1}{2}\delta_1^2 \right) (a) = \Omega a - a \Omega, \quad \forall a \in \mathcal{A}; \quad (3.1)$$

(2) *due to the commutativity of the δ_k 's,*

$$\delta_k(\Omega) = 0, \quad \forall k \in \mathbb{N}. \quad (3.2)$$

We point out that this \mathcal{H}_1 projectivity structure is a generalization of the projective structure on elliptic curves. On an elliptic curve, a projective structure means a choice of atlas such that the transition function between different charts can be chosen to be in $SL_2(\mathbb{R})$. The Ramanujan differential appears as a connection associated to such a projective structure.

A crucial observation of Connes and Moscovici [14] is that the extended Rankin-Cohen brackets on the modular Hecke algebra $\mathcal{A}(\Gamma)$ can be represented using elements in \mathcal{H}_1 and the element $\omega_4 \in \mathcal{A}(\Gamma)$. For example, the first Rankin-Cohen bracket can be realized by

$$RC_1(a \otimes b) = m((X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y)(a \otimes b)),$$

where $m : \mathcal{A}(\Gamma) \otimes \mathcal{A}(\Gamma) \rightarrow \mathcal{A}(\Gamma)$ is the multiplication on $\mathcal{A}(\Gamma)$. Generalizing the above formula of RC_1 , we define

$$\begin{aligned} A_{n+1} &:= S(X) A_n - n \Omega^o \left(Y - \frac{n-1}{2} \right) A_{n-1}, \\ B_{n+1} &:= X B_n - n \Omega \left(Y - \frac{n-1}{2} \right) B_{n-1}, \end{aligned}$$

where $A_{-1} := 0$, $A_0 := 1$ and $B_0 := 1$, $B_1 := X$, and Ω^o is the multiplication operator from the right by Ω . We notice that this is very similar to the sequences Eq. (2.2) of modular forms that Zagier used to reformulate the classical Rankin-Cohen brackets with $\Omega = \omega_4$. In general, the n -th Rankin-Cohen bracket can be written as

$$RC_n(a, b) := \sum_{k=0}^n \frac{A_k}{k!} (2Y + k)_{n-k}(a) \frac{B_{n-k}}{(n-k)!} (2Y + n - k)_k(b).$$

We remark that the above RC_n is a generalization of the classical n -th Rankin-Cohen bracket in the sense that when we take the modular Hecke algebra and we restrict the n -th Rankin-Cohen bracket on modular forms, we get the classical n -th Rankin-Cohen bracket. The advantage of this generalization is that now we can apply RC_n to an arbitrary algebra A on which \mathcal{H}_1 acts with a projective structure.

By a technique called *full injectivity*, Connes and Moscovici [14] proved their main theorem which states that the associative product can ultimately be lifted to a universal deformation formula for projective actions of the Hopf algebra \mathcal{H}_1 :

Theorem 3.3. ([14]) *The functor $RC_* := \sum RC_n$ applied to any algebra \mathcal{A} endowed with a projective structure yields a family of formal associative deformations of \mathcal{A} , whose products are given by*

$$f \star g = \sum RC_n(f, g) \hbar^n.$$

The full injectivity method used in the proof of the above theorem essentially says that there are enough different actions of \mathcal{H}_1 with projective structures so that any cocycle properties which lead to an associative deformation (on the algebras on which \mathcal{H}_1 acts) can always be lifted to the Hopf algebra level.

4. RANKIN-COHEN DEFORMATION VIA FEDOSOV

In the previous section, we have seen a beautiful result of Connes and Moscovici extending structures in modular form theory to study the Hopf algebra \mathcal{H}_1 . In this section, we look at the universal deformation formula obtained in Theorem 3.3 from the view point of transverse geometry of codimension one foliation.

In geometry, deformation of the algebra of smooth functions on a manifold has been studied for a long time. In particular, it is not difficult to see that the first order limit of an associative deformation of the commutative algebra of smooth functions on a manifold defines a Poisson bracket on the manifold. Here, by a Poisson bracket on a manifold P we mean a bilinear map $\{ , \} : C^\infty(P) \otimes C^\infty(P) \rightarrow C^\infty(P)$ such that for any $f, g, h \in C^\infty(P)$,

$$(1) \{f, g\} = -\{g, f\},$$

- (2) $\{f, gh\} = g\{f, h\} + \{f, g\}h,$
- (3) $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0.$

When P has a symplectic structure, namely, a non-degenerate closed 2-form ω on P , then the inverse of ω defines a Poisson bracket on $C^\infty(P)$ by $\{f, g\} := \omega^{-1}(df, dg)$. (ω is viewed as a skew symmetric bilinear form on TM , and its inverse defines a skew symmetric bilinear form on T^*M .) In mathematical physics, the phase space of a physical system is usually described by a symplectic manifold, and observables of a physical system are smooth functions on the symplectic manifold. Bayer-Flato-Fronsdal-Lichnerowicz-Sternheimer [1] pointed out that we can use deformations of the algebra of smooth functions on a symplectic (Poisson) manifold, the phase space of a physical system, to study the corresponding quantum system. They call a deformation of the algebra $C^\infty(P)$ with the first order term equal to the Poisson structure a *deformation quantization* of the symplectic (Poisson) manifold. An easy and beautiful example of such a theory is that in quantizing a free particle on \mathbb{R} , we have the algebra of quantum observables are generated by the position operator \hat{q} (the multiplication operator by function q) and the momentum operator $i\hbar \frac{d}{dx}$ on $L^2(\mathbb{R})$. Such an algebra is a deformation of the algebra of smooth functions on \mathbb{R}^2 (the corresponding phase space) with the standard symplectic structure $dp \wedge dq$. The product of this deformation quantization of $C^\infty(\mathbb{R}^2)$ can be written as

$$f \star g(x) = \exp\left(-\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)f(y)g(z)|_{x=y=z}, \quad f, g \in C^\infty(\mathbb{R}^2), \quad (4.1)$$

which is called the Moyal product [22]. With this geometry and physics in mind, we ask ourselves whether there is a geometric (maybe physical) interpretation of the Rankin-Cohen deformation.

We start with looking for a better understanding of a simplified version of the Rankin-Cohen deformation obtained in Theorem 3.3. We notice that if we set all δ_i ($i = 1, \dots, \infty$) and Ω (the projective structure) to be zero in the Rankin-Cohen bracket RC_n , we obtain a Universal Deformation Formula(UDF) of the 2-dimensional solvable Lie algebra h_1 associated to the $ax + b$ group, i.e. $h_1 = \text{span}\{X, Y\}$ with $[Y, X] = X$. We call the simplified deformation the *reduced Rankin-Cohen product*:

$$RC_{red} = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{k=0}^n \left[(-1)^k \binom{n}{k} X^k (2Y + k)_{n-k} \otimes X^{n-k} (2Y + n - k)_k \right], \quad (4.2)$$

with $Y_k = Y(Y + 1) \cdots (Y + k - 1)$.

The $ax + b$ acts on the upper half plane by translation, $(x, y) \mapsto (ax + b, ay)$. This induces an action of h_1 on $C^\infty(\mathbb{R} \times \mathbb{R}^+)$. Now applying the reduced Rankin-Cohen bracket (4.2) on $C^\infty(\mathbb{R} \times \mathbb{R}^+)$, we obtain a deformation of the algebra $C^\infty(\mathbb{R} \times \mathbb{R}^+)$. By using an argument via the orbit method (more precisely by a theorem of Gutt [19]), we together with Bieliavsky [2] proved that the reduced Rankin-Cohen deformation on $C^\infty(\mathbb{R} \times \mathbb{R}^+)$ is isomorphic to the Moyal product (4.1) over the half-plane.

In mid 90's, Giaquinto and Zhang proposed another UDF for h_1 (cf. [20]):

$$F = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{r=0}^n (-1)^r \binom{n}{r} X^{n-r} Y_r \otimes X^r Y_{n-r}.$$

With the same idea, we [2] were able to prove that the Giaquinto-Zhang UDF applied to $C^\infty(\mathbb{R} \times \mathbb{R}^+)$ is also isomorphic to Moyal product as well. We remark that the action of the universal enveloping algebra of h_1 on $C^\infty(\mathbb{R} \times \mathbb{R}^+)$ is fully injective. Therefore, the associativity of the Moyal product can be used to prove that the realization of the reduced Rankin-Cohen deformation on $C^\infty(\mathbb{R} \times \mathbb{R}^+)$ is associative. By the full injectivity assumption, we can give a new

proof that the reduced Rankin-Cohen deformation and also the Giaquinto-Zhang deformation are associative.

The above study provides a connection between the reduced Rankin-Cohen brackets and the well-known Moyal product. This also inspires us to pursue further whether we can give a geometric reconstruction of Connes-Moscovici's Rankin-Cohen deformation of the Hopf algebra \mathcal{H}_1 .

When asking ourselves this question, we realized that there is already a potential direction for the answer. Remember \mathcal{H}_1 was introduced in the study of transverse geometry of codimension one foliation. In particular, \mathcal{H}_1 acts on the smooth foliation algebra \mathcal{A}_Γ . In noncommutative geometry, \mathcal{A}_Γ is viewed as the algebra of smooth functions on the quotient space F^+X/Γ , which is usually a non-hausdorff topological space. A key observation here is that the volume form $dx \wedge dy/y^2$ defines a symplectic form on F^+X , which is invariant under the action of Γ . This leads us to the consideration of deformation quantization of F^+X with the symmetry of the pseudogroup Γ .

As a first step, we need to understand the first order term of the Rankin-Cohen deformation. As we have mentioned, when we consider the deformation of $C^\infty(P)$, the first order term is a Poisson structure. Now, for a noncommutative algebra \mathcal{A}_Γ , what is the first order term of a deformation? The answer to this question is that, it is a noncommutative Poisson structure.

A noncommutative Poisson structure on an algebra A is a degree 2 Hochschild cocycle Π such that the Gerstenhaber bracket $[\Pi, \Pi]_G$ is a coboundary.

We are able to prove the following result by a long but direct computation.

Theorem 4.1. ([2]) *Let A be an algebra equipped with an \mathcal{H}_1 action. Then $\Pi(a, b) = X(a)Y(b) - Y(a)X(b) + \delta_1(Y(a))Y(b) = m(RC_1(a \otimes b))$ defines a noncommutative Poisson structure on A .*

In the second step, we aim to understand the geometric meaning of a projective structure. As we mentioned, classically a projective atlas on a riemann surface X assign a principal $SL_2(\mathbb{R})$ bundle on X . In the case of codimension 1 foliation, we obtained the following theorem giving a geometric interpretation of a projective structure on \mathcal{A}_Γ analogous to this classical picture.

Theorem 4.2. ([2]) *In the standard action of \mathcal{H}_1 on a smooth foliation algebra \mathcal{A}_Γ , the projective structure is equivalent to the existence of an invariant connection on F^+X of the form*

$$\begin{aligned} \nabla_{\partial_x} \partial_x &= \mu(x, y) \partial_y, & \nabla_{\partial_x} \partial_y &= \frac{1}{2y} \partial_x, \\ \nabla_{\partial_y} \partial_x &= \frac{1}{2y} \partial_x, & \nabla_{\partial_y} \partial_y &= -\frac{1}{2y} \partial_y. \end{aligned}$$

We point out that the connection introduced in Theorem 4.2 is a symplectic torsion free connection on F^+X . This reminds us Fedosov's theory [18] of deformation quantization of a symplectic manifold.

We briefly explain Fedosov's construction of a deformation quantization on the manifold (F^+X, ω) . We consider a Weyl algebra bundle W whose fiber at every point p of F^+X is the Weyl algebra W_p consisting of formal power series

$$a(u, \hbar) = \sum_{k, |\alpha| \geq 0} \hbar^k a_{k, \alpha} u^\alpha.$$

Here \hbar is the formal parameter, $y = (u^1, u^2) \in T_x F^+X$, and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, and $y^\alpha = (u^1)^{\alpha_1} (u^2)^{\alpha_2}$.

The product on W_p is the Moyal product, for $a, b \in W_p$

$$\begin{aligned} a \circ b &= \exp \left(-\frac{i\hbar}{2} \omega^{ij} \frac{\partial}{\partial v^i} \frac{\partial}{\partial w^j} \right) a(v, \hbar) b(w, \hbar) |_{v=w=u} \\ &= \sum_{k=0}^{\infty} \left(-\frac{i\hbar}{2} \right)^k \frac{1}{k!} \omega^{i_1 j_1} \dots \omega^{i_k j_k} \frac{\partial^k a}{\partial u^{i_1} \dots \partial u^{i_k}} \frac{\partial^k b}{\partial u^{j_1} \dots \partial u^{j_k}}. \end{aligned}$$

Let \bigwedge be the bundle $\wedge^\bullet T^* F^+ X$. An abelian connection $D : \Gamma^\infty(W \otimes \bigwedge) \rightarrow \Gamma^\infty(W \otimes \bigwedge)$ is a connection on W with $D^2 a = 0$, for any $a \in \Gamma^\infty(W \otimes \bigwedge)$. When considering $W_D := \{a, Da = 0\}$, Fedosov proved that

Theorem. ([18]) *For all $a_0 \in C^\infty(F^+ X)[[\hbar]]$, there exists a unique section $a \in W_D$, noted as $\sigma^{-1}(a_0)$, such that $\sigma(a) := a(x, 0, \hbar) = a_0$. Hence, σ is a bijection between W_D and $C^\infty(F^+ X)[[\hbar]]$. And we can define on $C^\infty(F^+ X)[[\hbar]]$ an associative product*

$$a \star b = \sigma(\sigma^{-1}(a) \circ \sigma^{-1}(b)).$$

In the case that $F^+ X$ has a Γ -invariant symplectic connection, the second named author proved [27] that the above deformation quantization of $C_c^\infty(F^+ X)$ defines a deformation of the corresponding crossed product algebra $\mathcal{A}_\Gamma = C_c^\infty(F^+ X) \rtimes \Gamma$.

When we are given the Γ -invariant symplectic torsion free connection ∇ (Thm. 4.2) on $F^+ X$, if we assume furthermore that $\mu(x, y)$ is of the form $y\nu(x)$ with $\nu(x)$ an arbitrary smooth function of variable x , the connection is actually flat. This allows to find an explicit formula for the abelian connection D . The equation $Da = 0$ gives us a system of differential equations. And starting from $a_{0,0} = f$ and solving the system by induction, we obtain two sequences of elements:

$$A_{m+1} = -XA_m - m \frac{\mu}{y^3} \left(Y - \frac{m-1}{2} \right) A_{m-1}, \quad (4.3)$$

$$B_{m+1} = XB_m - m \frac{\mu}{y^3} \left(Y - \frac{m-1}{2} \right) B_{m-1}. \quad (4.4)$$

where $X = \frac{1}{y} \frac{\partial}{\partial x}$, $Y = -y \frac{\partial}{\partial y}$. Extending the above formulas onto the smooth foliation algebra \mathcal{A}_Γ , we obtained the same recurrence relation that Connes and Moscovici used in their definition of generalized Rankin-Cohen brackets, so one can consider what we get as another realization of the Rankin-Cohen deformation. The advantage of this argument is that one gets the associativity without any extra effort. We show [2] that in this context we can also prove the “full injectivity”, which allows us to conclude the associativity of the Rankin-Cohen deformation at the Hopf algebra level.

We notice that in Thm 4.1, the first Rankin-Cohen bracket always defines a Poisson structure no matter whether there is an \mathcal{H}_1 projective structure or not. This inspires to ask whether \mathcal{H}_1 has a Universal Deformation Formula without assuming the existence of a projective structure. In the language of deformation quantization, this question is whether there is a deformation of the algebra \mathcal{A}_Γ without the existence of an invariant symplectic connection. We learned from Fedosov [18] and also Gorokhovskiy-Bressler-Nest-Tsygan [1] that the existence of an invariant symplectic torsion free connection is not a necessary condition for a smooth foliation algebra to have a deformation quantization. Actually, we can always construct a deformation quantization of a smooth foliation algebra \mathcal{A}_Γ using the idea of algebroid stacks. With this in mind, we proved the following theorem, where we can drop the assumption of the existence of a projective structure.

Theorem 4.3. ([28]) *The Hopf algebra \mathcal{H}_1 has a universal deformation formula, i.e. there is an element $R \in \mathcal{H}_1[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} \mathcal{H}_1[[\hbar]]$ satisfying*

$$((\Delta \otimes 1)R)(R \otimes 1) = ((1 \otimes \Delta)R)(1 \otimes R), \quad (4.5)$$

$$(\epsilon \otimes 1)(R) = 1 \otimes 1 = (1 \otimes \epsilon)(R). \quad (4.6)$$

As a side remark, we also obtained [28] a proof of the associativity of the Eholzer product of a reasonable length using some elementary methods.

5. RANKIN-COHEN VIA REPRESENTATIONS

Rankin-Cohen brackets and related deformation questions can also be studied using the theory of infinite dimensional representations of $SL_2(\mathbb{R})$. First of all, one can explicitly give an interpretation of these brackets using unitary representation theory of $SL_2(\mathbb{R})$. This fact is known by experts for a long time³ but does not seem to be clearly written anywhere. The main result is the following theorem:

Theorem. *Let $f \in \mathcal{M}_{2k}, g \in \mathcal{M}_{2l}$ be two modular forms. Let $\pi_f \cong \pi_{\deg f}, \pi_g \cong \pi_{\deg g}$ be associate representations which are discrete series of the group $SL_2(\mathbb{R})$. The tensor product of these two representations is decomposed into a direct sum of discrete series,*

$$\pi_f \otimes \pi_g = \bigoplus_{n=0} \pi_{\deg f + \deg g + 2n}.$$

The Rankin-Cohen bracket $[f, g]_n$ gives (up to a scalar) the minimal K -weight vectors in the representation space of the component $\pi_{\deg f + \deg g + 2n}$.

We remark that the result about tensor product between $SL_2(\mathbb{R})$ representations are well-known (c.f. [26] by Repka). The new part in the above theorem is the relation between Rankin-Cohen brackets and representation theory.

The representation π_f is constructed in the following way: let $f \in \mathcal{M}_{2k}(\Gamma)$ be a modular form, one can associated to it a function over $\Gamma \backslash SL_2(\mathbb{R})$ by using the following mapping:

$$(\sigma_{2k}f)(\gamma) = f|_k \gamma(i) = (ci + d)^{-2k} f\left(\frac{ai + b}{ci + d}\right), \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$

This function belongs to

$$C^\infty(\Gamma \backslash SL_2(\mathbb{R}), 2k) = \{F \in C^\infty(\Gamma \backslash SL_2(\mathbb{R})), F(\gamma \cdot r_\theta) = \exp(i2k\theta)F(\gamma)\},$$

where

$$r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

By taking into account the natural action of $SL_2(\mathbb{R})$ on $C^\infty(\Gamma \backslash SL_2(\mathbb{R}))$:

$$(\pi(h)F)(g) = F(gh),$$

one gets a representation of $SL_2(\mathbb{R})$ and also a representation of the complexified Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ by taking the smallest invariant subspace which contains the orbit of $\sigma_{2k}f$. One can show that this representation is a discrete series of weight $2k$. In this step, we take all the

³For example, Deligne wrote in 1973 (cf. [15]) : Remarque 2.1.4. L'espace $F(G, GL_2(\mathbb{Z}))$ ci-dessus est stable par produit. D'autre part, $D_{k-1} \otimes D_{l-1}$ contient les $D_{k+l+2m-1}$ ($m \geq 0$). Pour $m = 0$, ceci correspond au fait que le produit fg d'une forme modulaire holomorphe de poids k par une de poids l , en est une de poids $k+l$. Pour $m = 1$, en coordonnées (1.1.5.2), on trouve que $l \frac{\partial f}{\partial z} \cdot g - kf \cdot \frac{\partial g}{\partial z}$ est modulaire holomorphe de poids $k+l+2$, et ainsi de suite. De même dans le cadre adélique.

vectors in a base of the representation space to a subspace of $C^\infty(\mathbb{H})$, by using the inverse of $\sigma_{2(k+n)}$, $n \geq 0$.

With this interpretation, we studied the deformed products in a more general setting. From now on we drop the holomorphy condition on a modular form, and consider all functions which satisfy the modularity condition with respect to Γ . We denote the algebra under consideration by $\widetilde{\mathcal{M}}(\Gamma)$. Enlarging $\widetilde{\mathcal{M}}(\Gamma)$, we consider the tensor algebra

$$\widetilde{\mathcal{M}}(\Gamma)^\otimes = \sum_n \widetilde{\mathcal{M}}(\Gamma)^{\otimes n},$$

and define

$$\begin{aligned} \mathbf{M} : \widetilde{\mathcal{M}}(\Gamma)^\otimes &\rightarrow \widetilde{\mathcal{M}}(\Gamma), \\ f_1 \otimes f_2 \otimes \cdots \otimes f_n &\mapsto f_1 f_2 \cdots f_n, \end{aligned} \quad (5.1)$$

and we extend the degree operator by

$$\deg(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \sum_{i=1}^n \deg f_i.$$

We define a derivation on $\widetilde{\mathcal{M}}(\Gamma)$, for an element $f \in \widetilde{\mathcal{M}}(\Gamma)$ of degree $2k$,

$$\tilde{X}f = \frac{1}{2\pi i} \frac{df}{dz} - \frac{2kf}{4\pi \operatorname{Im}(z)}. \quad (5.2)$$

We extend \tilde{X} to $\widetilde{\mathcal{M}}(\Gamma)^\otimes$ by the Leibnitz rule

$$\tilde{X}(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \sum_{i=1}^n f_1 \otimes f_2 \otimes \cdots \otimes \tilde{X}f_i \otimes \cdots \otimes f_n.$$

Inspired by the Rankin-Cohen brackets (2.1), we study the following two families of formal products. The first family is $\star : \widetilde{\mathcal{M}}(\Gamma)^\otimes[[\hbar]] \otimes \widetilde{\mathcal{M}}(\Gamma)^\otimes[[\hbar]] \rightarrow \widetilde{\mathcal{M}}(\Gamma)^\otimes[[\hbar]]$ defined by linear extension and the following formula:

$$f \star g = \sum \frac{A_n(\deg f, \deg g)}{(\deg f)_n (\deg g)_n} \left(\sum_{r=0}^n (-1)^r \tilde{X}^r \binom{\deg f + n - 1}{n - r} f \otimes \tilde{X}^{n-r} \binom{\deg g + n - 1}{r} g \right) \hbar^n, \quad (5.3)$$

where $f, g \in \widetilde{\mathcal{M}}(\Gamma)^\otimes$. The second family of products $*$ is the restriction of \star defined by Eq. (5.3) to $\widetilde{\mathcal{M}}(\Gamma) \subset \widetilde{\mathcal{M}}(\Gamma)^\otimes$ composed with the application \mathbf{M} defined in (5.1). More concretely, $* : \widetilde{\mathcal{M}}(\Gamma)[[\hbar]] \times \widetilde{\mathcal{M}}(\Gamma)[[\hbar]] \rightarrow \widetilde{\mathcal{M}}(\Gamma)[[\hbar]]$ is defined as follows:

$$\begin{aligned} f * g &= \mathbf{M}(f \star g) \\ &= \sum \frac{A_n(\deg f, \deg g)}{(\deg f)_n (\deg g)_n} \left(\sum_{r=0}^n (-1)^r \tilde{X}^r \binom{2k + n - 1}{n - r} f \tilde{X}^{n-r} \binom{2l + n - 1}{r} g \right) \hbar^n \\ &= \sum \frac{A_n(\deg f, \deg g)}{(\deg f)_n (\deg g)_n} [f, g]_n \hbar^n, \end{aligned} \quad (5.4)$$

where $f, g \in \widetilde{\mathcal{M}}$, and the notation $(\alpha)_n := \alpha(\alpha + 1) \cdots (\alpha + n - 1)$. We will always take the (natural) assumption $A_0 = 1$ and $A_1(x, y) = xy$, and A_i ($i \geq 2$) is a polynomial of two variables.

Our main goal is to study the associativity of the above two family of products. We name the associativity of the product \star *strong associativity*, and the associativity of $*$ *weak associativity*.

The third named author proved [31] that, while it is obvious that a family of coefficient functions $A_n(\deg f, \deg g)$ which defines a strongly associative product also defines also a weakly associative product, the inverse, which is not clear *a priori*, is also true. This equivalence between the two types of associativity is used to show the following results,

Theorem 5.1. ([31]) *Cohen-Manin-Zagier [6] have found all associative formal deformation $*$: $\widetilde{\mathcal{M}}[[\hbar]] \times \widetilde{\mathcal{M}}[[\hbar]] \rightarrow \widetilde{\mathcal{M}}[[\hbar]]$ defined by the linearity and the formula*

$$f * g = \sum \frac{A_n(\deg f, \deg g)}{(\deg f)_n (\deg g)_n} [f, g]_n \hbar^n, \quad (5.5)$$

where $\widetilde{\mathcal{M}}(\Gamma)$ is the space of the functions which satisfy the modularity condition, and the notation $(\alpha)_n := \alpha(\alpha+1) \cdots (\alpha+n-1)$. One assumes again $A_0 = 1$ and $A_1(x, y) = xy$.

The main reason for this claim to be valid is that the determinant of some 2×2 linear system is non-zero, and the drop of holomorphy allows us to have the total freedom to modify a function in the interior of a fundamental domain. Furthermore, with the help of some computation of certain multivariable polynomials done by Mathematica, the third named author was able to prove the following proposition.

Proposition 5.2. ([31]) *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ such that $\mathcal{M}(\Gamma)$ admits the unique factorization property (for example $SL_2(\mathbb{Z})$ itself), let $F_1, F_2, G_1, G_2 \in \mathcal{M}(\Gamma)$ such that*

$$RC(F_1, G_1) = RC(F_2, G_2),$$

as formal series in $\mathcal{M}(\Gamma)[[\hbar]]$, then there exists a constant C such that

$$F_1 = CF_2, G_2 = CG_1.$$

6. HANKEL FORMS AND A NEW \mathcal{H}_1 ACTION

In this section, we aim to set up a connection between the Hopf algebra \mathcal{H}_1 and the theory of high order Hankel forms introduced by Janson and Peetre⁴ [21]. We hope that such a connection will inspire more interactions among transverse geometry, number theory, and harmonic analysis. Similar to what we did in Sec. 5, we will consider a set of functions more general than modular forms. Instead of dropping the holomorphy property of a modular form, in this section we will drop the modularity property. For example, we will consider holomorphic functions on the unit disk of \mathbb{C} that are square integrable with respect to some measure.

Let D be the unit disk in the complex plane \mathbb{C} . Let $\eta(z) = 1 - |z|^2$, and $d\sigma(z) = (1/\pi)dx dy$. The weighted Bergman space $A^{2\alpha}(D)$ ($\alpha > -1$) is defined to be

$$\{f : \bar{\partial}f = 0, \int_D |f(z)|^2 \eta^\alpha d\sigma(z) < \infty\}.$$

The group $SL(2, \mathbb{R})$ acts on the weighted Bergman space $A^{2\alpha}(D)$ as follows, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\pi_{2k}(\gamma)(f)(z) = f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-\alpha}. \quad (6.1)$$

It is easy to check that above action π_{2k} defines a unitary representation of $SL(2, \mathbb{R})$, which is actually irreducible. Such a representation is called a discrete series of $SL(2, \mathbb{R})$. Applying the result of Repka [26], a tensor product of representations in the discrete series is a direct

⁴In literature, there is a huge amount of study of Hankel forms and transvectants. Due to our limited knowledge, we have only cited those references related to our work.

sum of irreducible $SL(2, \mathbb{R})$ representations in the discrete series. More explicitly, as $SL(2, \mathbb{R})$ representations, we have the following decomposition,

$$\pi_{2k} \otimes \pi_{2l} = \bigoplus_{i \geq 0} \pi_{2k+2l+2i}.$$

We denote the projection from $A^{2k}(D) \otimes A^{2l}(D)$ to $A^{2k+2l+2i}$ by Π_i , which is $SL_2(\mathbb{R})$ equivariant bilinear map from $A^{2k}(D) \times A^{2l}(D)$ to $A^{2k+2l+2i}$. Such an equivariant bilinear form is called by Janson and Peetre [21] a Hankel form of weight $2i$. Similar to the Rankin-Cohen brackets, P_{2i} can be expressed using the derivatives and the weights of the two components, i.e. up to a constant,

$$\Pi_{2i}(f, g)(z) = \sum_{s=0}^i (-1)^{i-s} \binom{i}{s} \frac{1}{(w(f))_s (w(g))_{i-s}} \partial^s f \partial^{i-s} g,$$

where $w(f)$ and $w(g)$ are weights of f and g .

We remark that the above discussion of Hankel forms of high weights is closely related to the theory of transvectant. Olver-Sanders [23], Choi-Mourrain-Solé [4], and El Gradechi [17] provided an explicit relation between transvectants, Rankin-Cohen brackets, and Moyal product [23]. One can even use these interesting relations to give a proof of the associativity of the Rankin-Cohen deformation, which was explained by Pevzner [24]. We refer readers to the cited references for more details.

In the following of this section, we aim to introduce an algebra \mathcal{B}_Γ associated to a pseudogroup Γ of holomorphic transformations on a complex domain Σ of dimension one. And we will show that there is a natural action of \mathcal{H}_1 on \mathcal{B}_Γ . In the special case when Γ is trivial and Σ is D , the unit disk in \mathbb{C} , the Rankin-Cohen deformation gives rise to Hankel forms of higher weights.

Let Σ be a complex domain of dimension 1 (maybe of several components). Γ is a pseudogroup acting on Σ by holomorphic transformations. Let $T^{1,0}\Sigma$ be the holomorphic tangent bundle of Σ , which is equipped with a Γ action

$$\gamma(x, y) = (\gamma(x), \partial_x \gamma(x)y),$$

where (x, y) is a point on σ . We assume that the holomorphic tangent bundle is trivialized. Namely, we have chosen a global coordinates (x, y) on $T^{1,0}\Sigma$.

Remark 6.1. *We have required the bundle $T^{1,0}\Sigma$ to be trivialized in order to work with holomorphic functions and holomorphic vector fields on Σ . If $T^{1,0}\Sigma$ is not a trivial bundle over Σ , then we will have to work with sheaves of holomorphic functions and holomorphic vector fields. Accordingly, we will have to enlarge the notion of Hopf algebra to Hopf algebroid like in [12].*

We consider the space of holomorphic functions on $T^{1,0}\Sigma$, and denote it by $A_{T^{1,0}\Sigma}$. The action of Γ on $T^{1,0}\Sigma$ induces an action of Γ on $A_{T^{1,0}\Sigma}$. Therefore, we define \mathcal{B}_Γ to be the crossed product algebra $A_{T^{1,0}\Sigma} \rtimes \Gamma$.

We remark that when Σ is the unit disk in \mathbb{C} , the space of holomorphic functions on $T^{1,0}D$ has a natural decomposition,

$$A_{T^{1,0}D} = \bigoplus_{i \geq 0} A_D y^i,$$

where y is the coordinate along the fiber direction of $T^{1,0}D$, and A_D is the space of holomorphic functions on D . $SL_2(\mathbb{R})$, the group of holomorphic transformations of D , acts on each component $A_D y^i$ exactly like the $SL_2(\mathbb{R})$ action on the weighted Bergman space $A^{2i}(D)$.

Actually both A_D and $A^{2i}(D)$ share a dense subalgebra, the algebra of polynomials of variable x .

To introduce the Hopf algebra \mathcal{H}_1 action on B_Γ , we first introduce two holomorphic vector fields on $T^{1,0}\Sigma$. Define

$$X = y\partial_x, \quad Y = y\partial_y.$$

It is easy to check that for an element $\gamma \in \Gamma$,

$$U_\gamma X U_{\gamma^{-1}} = X - y \frac{\partial_x^2 \gamma^{-1}}{\partial_x \gamma^{-1}} Y, \quad U_\gamma Y U_{\gamma^{-1}} = Y,$$

where U_γ is the action of γ on $A_{T^{1,0}\Sigma}$.

The vector fields X and Y lift to act on the algebra \mathcal{B}_Γ by acting on the component of $A_{T^{1,0}\Sigma}$. We define $\delta_1 : \mathcal{B}_\Gamma \rightarrow \mathcal{B}_\Gamma$ by

$$\delta_1(fU_\gamma) = y\partial_x(\log(\partial_x \gamma^{-1}))fU_\gamma.$$

It is not difficult to check that operators X, Y, δ_1 satisfy

$$[Y, X] = X, \quad [Y, \delta_1] = \delta_1,$$

and $[X, \delta_1] = \delta_2$, where $\delta_2 : \mathcal{B}_\Gamma \rightarrow \mathcal{B}_\Gamma$ is an operator defined by

$$\delta_2(fU_\gamma) = y^2 \partial_x^2 (\log(\partial_x(\gamma^{-1}))).$$

Continuing this procedure, we have $\delta_n : \mathcal{B}_\Gamma \rightarrow \mathcal{B}_\Gamma$ by

$$\delta_n(fU_\gamma) = X(\delta_{n-1}(fU_\gamma)).$$

Proposition 6.2. *Connes-Moscovici's Hopf algebra \mathcal{H}_1 acts naturally on the algebra \mathcal{B}_Γ .*

Proof. The proof that \mathcal{H}_1 acts on \mathcal{B}_Γ is a repetition of Connes-Moscovici's proof that \mathcal{H}_1 acts on the smooth foliation algebra \mathcal{A}_Γ . Here we are replacing smooth functions by holomorphic functions and differentiations by holomorphic differentiations. But the Hopf algebraic structure involved does not change at all. \square

As an application of the Rankin-Cohen deformation, assuming Γ is trivial, we apply the reduced Rankin-Cohen deformation (4.2) to the algebra $A_{T^{1,0}D}$, we have for $f(x)y^k, g(x)y^l$,

$$RC_{red}(f(x)y^k, g(x)y^l) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{s=0}^n (-1)^s \left[\binom{n}{s} \frac{(2k+n-1)!}{(2k+s-1)!} \partial_x^s(f)(x) \frac{(2l+n-1)!}{(2l+n-s-1)!} \partial_x^{n-s}(g)(x) \right] y^{k+l+n}.$$

When we take the component of \hbar^n , then we find

$$R_n(fy^k, gy^l) = (-1)^n \frac{(2k+n-1)!(2l+n-1)}{((2k-1)!(2l-1)!)} \sum_{s=0}^n (-1)^{n-s} \left[\binom{n}{s} \frac{1}{(2k)_s (2l)_{n-s}} \partial_x^s(f)(x) \partial_x^{n-s}(g)(x) \right] y^{k+l+n}.$$

Recalling the relation between $A_{T^{1,0}D}$ and the weighted Bergman space $A^{2n}(D)$, we can extend the action of h_1 onto $\oplus_{n \geq 0} A^{2n}(D)$ and conclude with the following proposition.

Proposition 6.3. *The reduced i -th Reduced-Cohen bracket on $\oplus_{n \geq 0} A^{2n}(D)$ is a Hankel form of weight $2i$. The associativity of the reduced Rankin-Cohen deformation implies these family of Hankel forms defines an associative deformation of $\oplus_{n \geq 0} A^{2n}(D)$.*

We remark that in [29] (and also [16]), Unterberger-Unterberger introduced some operator symbol calculus associated to weighted Bergman spaces. And they are also able to construct an associative deformation on $\oplus_{n \geq 0} A^{2n}(D)$. It is an interesting question to compare our deformation in Proposition 6.3 and the deformation obtained in [29], while our coefficients is much simpler than the ones in [29].

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